Equivalence of post and prior sum rules for inclusive breakup reactions

M. Ichimura
Institute of Physics, University of Tokyo, College of General Education, Komaba 3-8-1, Tokyo 153, Japan

N. Austern and C. M. Vincent
Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260
(Received 28 January 1985)

A critical examination of sum rules derived previously by Austern and Vincent (post form) and by Udagawa and Tamura (prior form) demonstrates that agreement between the two approaches is obtained if certain approximations implicit in the Udagawa-Tamura prior-form derivation are avoided. We examine the relation of the two approaches to singularities of the post-form distorted wave Born approximation matrix element and to the procedures for reduction of a many-body theory by use of effective operators in a model space. The two-step heuristic model is seen to be invalid for prior-form inelastic breakup; it is necessary to take account of nuclear excitations during projectile breakup. Careful treatment of the non-Hermiticity of kinetic energy operators with respect to continuum wave functions is required.

I. INTRODUCTION

Disagreements between recent papers about inclusive breakup raise interesting questions of understanding of basic reaction theory. Inclusive breakup refers to reactions of the type

$$a + A \rightarrow b + \text{anything} ,$$

where b is a definite fragment of the incident projectile,

$$a = b + x ,$$

and the energy of b in the exit channel is low enough so the remaining dynamical system $x + A$ is unbound. Thus an inclusive experiment sums over exit channels that have three or more bodies in the continuum. In each such channel the final condition of the system $x + A$ may have x and A in their ground states [elastic breakup (EB)] or in excited states (inelastic breakup), or there may be particle transfer between x and A. In the work of Li, Udagawa, and Tamura[1] (LUT) the sum of breakup events other than elastic breakup is called breakup fusion (BF).

Direct reaction analyses of inclusive breakup treat b as a spectator to the subsequent dynamics of the $x + A$ system. In such "two-step" approaches completeness allows reduction of the sum over channels to a closed form operator expression, a typical quantum mechanical sum rule. LUT apply completeness to the prior-form derivation of the breakup cross section. They criticize a related sum rule expression obtained by Austern and Vincent[2] (AV) and Kasano and Ichimura[3] (KI), who apply completeness to the post-form derivation. The LUT criticism is extended to papers by Baur and Trautmann and collaborators,[4] whose calculational method[5] can be regarded as an approximation of the AV-KI theory.

Both sum rules take the form of ground state expectations of the Green's function G_{xA} for the interaction of the unobserved particle x with the target nucleus, in combination with various optical potential interactions. Since the sum rules concern sums over alternative modes of decay of $x + A$, their formal derivation requires consideration of the internal coordinates of the nuclei. But these coordinates no longer appear in the final expressions with optical potentials. Hence an essential step in the derivation of the sum rules is the "optical reduction," which suppresses internal coordinates.

To obtain consistent post and prior derivations of sum rules, it is necessary that both approaches start from the same many-body dynamical model and that they employ comparable optical reductions. We show in Sec. IV that an exact optical reduction in the derivation of the prior-form sum rule is rather complicated, and it does not agree with the expression discussed by LUT, which we show is based on an approximation. From the numerical calculations given by LUT, it is clear that the discrepancy between the exact and approximate prior-form sum rules is large. The corresponding optical reduction in the post-form derivation in Sec. II is easier, because the post interaction V_{bx} does not contain the coordinates of A. We show in Sec. IV that the corrected prior-form sum rule transforms correctly into the AV post-form sum rule; our transformation formally resembles one given by LUT. Thus the AV post-form sum rule is correct, and the LUT criticisms originate from approximations in their optical reduction.

The post-form derivation is not without its own difficulties. Even on the energy shell the post-form breakup matrix element needs a convergence factor[6,7] to correctly suppress long range oscillations in the integration. Off the energy shell the disconnected structure of the matrix element causes a delta function divergence, and this is potentially more troublesome, because the derivation of the sum rule requires some use of off-energy-shell matrix elements. We reexamine the post-form derivation in Sec. II, and we show that the off-energy-shell divergence of the matrix elements does not damage the sum rule.

The oscillatory integrals and off-energy-shell divergence

of the post-form matrix element are examined in detail in Sec. III, by a systematic transformation from the fully connected prior-form matrix element. It is seen that these difficulties originate from the non-Hermiticity of kinetic energy operators with respect to continuum wave functions.

All our analysis is based on one of the methods of AV, the "operator derivation," which explicitly sums over exit channels of the $x+A$ system. The AV "wave function derivation" is not adapted to the formal mathematics used in Sec. II and it seems entirely inapplicable to the complications of the prior-form discussion. However, it is interesting that the flux conservation equation that is central to the wave function derivation is equivalent to an operator identity for optical Green's functions, introduced by KI and LUT to separate elastic breakup from breakup fusion. This identity appears in Sec. II. Conclusions are discussed in Sec. V.

II. POST-FORM SUM RULE

In this section we review the AV operator derivation, with greater emphasis on the mathematical details that justify the manipulations used. As before we treat a model Hamiltonian

$$H=H_A(\xi)+K_b+K_x+V_{xA}+U_b+V_{bx} , \tag{2.1}$$

in which the interaction V_{bA} of the spectator particle with the target nucleus has been replaced by the optical potential U_b. Thus it is assumed that interactions with the spectator particle do not cause excitations of the target nucleus, in agreement with the two-step reaction mechanism emphasized by LUT. In our model $H_A(\xi)$ is the internal Hamiltonian of the target nucleus, K_b and K_x are kinetic energy operators, and V_{bA}, V_{xA}, and V_{bx} are interaction potentials—ordinarily each of these potentials will be a sum of nucleon-nucleon interactions. For simplicity the target nucleus is assumed to be so massive that it remains at rest at the origin. In addition, since explicit wave functions and Hamiltonians for the internal states of b and x are not needed, for simplicity they are not included in our notation. However, the target nucleus ground state wave function is written as Φ_A, with energy E_A, so

$$H_A\Phi_A=E_A\Phi_A . \tag{2.2}$$

The eigenstates of the system $x+A$ are written Ψ^c_{xA}, and they satisfy

$$H_{xA}\Psi^c_{xA}=E^c\Psi^c_{xA} , \tag{2.3}$$

with

$$H_{xA}\equiv H_A+K_x+V_{xA} . \tag{2.4}$$

The internal eigenstate of the projectile is $\phi_a(\mathbf{r}_{bx})$, with energy E_a, where $\mathbf{r}_{bx}=\mathbf{r}_b-\mathbf{r}_x$.

A general post-form DWBA expression for inclusive breakup to a range of outgoing momenta $\Delta\mathbf{p}$ is

$$\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\text{post}}=\frac{(2\pi)^4}{v_a}\sum_c |\langle\chi_b^{(-)}\Psi^c_{xA}\,|\,V_{bx}\,|\,\chi_a^{(+)}\phi_a\Phi_A\rangle|^2 \times\delta(E-E_b-E^c) , \tag{2.5}$$

for total energy E. Here we use optical wave functions defined by

$$[K_a+U_a(r_a)]\chi_a^{(+)}(\mathbf{r}_a)=(E-E_A-E_a)\chi_a^{(+)}(\mathbf{r}_a) , \tag{2.6}$$

$$[K_b+U_b^\dagger(r_b)]\chi_b^{(-)}(\mathbf{r}_b)=E_b\chi_b^{(-)}(\mathbf{r}_b) . \tag{2.7}$$

The post-form residual interaction in (2.5) takes the simple, Hermitian form V_{bx}, both because other interactions in H are included in the definition of the exit channel wave functions, and because of the choice U_b for the spectator-nucleus interaction. Use of the more fundamental interaction V_{bA} in (2.1) instead of U_b would require an additional term $V_{bA}-U_b$ in the residual interaction; with this term present an orderly sum rule would be unlikely.

A. Sum rule derivation

The inclusive sum on c is performed by the steps

$$\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\text{post}}=\frac{-(2\pi)^4}{\pi v_a}\,\text{Im}\sum_c\langle\chi_a^{(+)}\phi_a\Phi_A\,|\,V_{bx}\,|\,\chi_b^{(-)}\Psi^c_{xA}\rangle(E^+-E_b-E^c)^{-1}\langle\Psi^c_{xA}\chi_b^{(-)}\,|\,V_{bx}\,|\,\chi_a^{(+)}\phi_a\Phi_A\rangle , \tag{2.8}$$

$$=\frac{-(2\pi)^4}{\pi v_a}\,\text{Im}\langle\chi_a^{(+)}\phi_a\Phi_A\,|\,V_{bx}\,|\,\chi_b^{(-)}\rangle(E^+-E_b-H_{xA})^{-1}\langle\chi_b^{(-)}\,|\,V_{bx}\,|\,\chi_a^{(+)}\phi_a\Phi_A\rangle , \tag{2.9}$$

where in (2.9) the energy denominator of (2.8) is replaced by its operator equivalent, using quantum mechanical completeness. Because V_{bx} and the optical wave functions in (2.9) do not depend on internal coordinates of the target nucleus, the Φ_A expectation in (2.9) allows an optical reduction of the Green's function

$$(\Phi_A\,|\,(E^+-E_b-H_{xA})^{-1}\,|\,\Phi_A)=(E_x^+-K_x-U_x)^{-1}\equiv G_x , \tag{2.10}$$

where $E_x\equiv E-E_b-E_A$ and U_x is the Feshbach formal optical potential[8] for particle x. Then

$$\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\text{post}}=\frac{-(2\pi)^4}{\pi v_a}\,\text{Im}\langle\rho_b(\mathbf{r}_x)\,|\,G_x\,|\,\rho_b(\mathbf{r}_x)\rangle , \tag{2.11}$$

with the source function

$$\rho_b(\mathbf{r}_x)\equiv(\chi_b^{(-)}\,|\,V_{bx}\,|\,\chi_a^{(+)}\phi_a\rangle . \tag{2.12}$$

The optical reduction (2.10) is a formally exact result. In a typical derivation[2,8] the product expression

$$Z(\xi,\mathbf{r}_x)\equiv(E^+-E_b-H_{xA})^{-1}\Phi_A(\xi)\rho(\mathbf{r}_x) \tag{2.13}$$

is converted into a differential equation

$$(E-E_b-H_{xA})Z=\Phi_A\rho\ . \tag{2.14}$$

By use of projection operators

$$P=|\Phi_A\rangle\langle\Phi_A|\ ,\quad Q=1-P\ , \tag{2.15}$$

the differential equation for Z is converted into coupled equations for PZ and QZ, which are solved to give

$$[E-E_b-PH_{xA}P - PH_{xA}Q(E^+-E_b-QH_{xA}Q)^{-1}QH_{xA}P]PZ=\Phi_A\rho\ . \tag{2.16}$$

The left-hand side of (2.16) defines an effective interaction U_x, which appears in (2.10) in the associated Green's function G_x. It is important to recognize that the operator U_x in the sum rule originates from this optical reduction.

A helpful transformation[1,3,9] of the Green's function in (2.11) is obtained from the adjoint pair of equations

$$G_x=G_0(1+U_xG_x) =(1+G_x^\dagger U_x^\dagger)G_0(1+U_xG_x)-G_x^\dagger U_x^\dagger G_x\ , \tag{2.17a}$$

$$G_x^\dagger=(1+G_x^\dagger U_x^\dagger)G_0^\dagger =(1+G_x^\dagger U_x^\dagger)G_0^\dagger(1+U_xG_x)-G_x^\dagger U_xG_x\ , \tag{2.17b}$$

in which G_0 is the free Green's function for particle x at energy E_x. By subtraction of these equations,

$$\mathrm{Im}G_x\equiv\frac{1}{2i}(G_x-G_x^\dagger) =(1+G_x^\dagger U_x^\dagger)\,\mathrm{Im}G_0(1+U_xG_x)+G_x^\dagger W_xG_x\ , \tag{2.18}$$

in which W_x is the imaginary part of U_x. The first term on the right-hand side (RHS) of (2.18) is a product of wave operators with incoming boundary conditions for the motion of particle x under the distorting potential $U_x^\dagger$, multiplied by an energy-conserving operator delta function,

$$\mathrm{Im}G_0=-\pi\delta(E_x-K_x)\ . \tag{2.19}$$

Upon substitution in (2.11), the first term of (2.18) yields[3] exactly the post-form DWBA total cross section for elastic breakup. Thus (2.11) becomes

$$\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\mathrm{post}}=\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\mathrm{post}}^{\mathrm{EB}}+\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\mathrm{post}}^{\mathrm{BF}}, \tag{2.20}$$

in which the "breakup fusion" cross section (see the Introduction) incorporates all processes other than elastic breakup. We see

$$\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\mathrm{post}}^{\mathrm{BF}}=\frac{-(2\pi)^4}{\pi v_a}\langle\rho_b|G_x^\dagger W_xG_x|\rho_b\rangle\ . \tag{2.21}$$

Equations (2.20) and (2.21) express the AV-KI result.

The transformation from (2.11) to (2.20) and (2.21) expresses the operator $\mathrm{Im}G_x$ in terms of the quantities W_x and $\mathrm{Im}G_0$, whose properties are known more fully. In particular, although W_x is the imaginary part of a formal optical potential, and it can be quite complicated, it is helpful in (2.21) that W_x is real and has finite range.

Transformation (2.18) has an interesting physical interpretation, in that the expectation of the two sides of this equation with respect to $|\rho_b\rangle$ gives the flux conservation equation (17) of the AV wave function derivation. Thus (2.18) separates the total breakup flux into its component parts. This separation is possible only because in the model discussed breakup takes place by an interaction V_{bx} that does not excite the target nucleus. The prior-form discussion is more complicated, as we see in Sec. IV.

B. Convergence questions

Several steps in the above derivation require more careful discussion. It was already noted by AV that (2.11) is the imaginary part of an expression whose real part is infinite. This divergence arises from a *disconnected part* of the post-form matrix element, which we discuss in detail in Sec. III. Transformation (2.18) isolates the disconnected diagram in the EB part of the cross section, so that it does not appear in (2.21).

The post-form breakup matrix element in (2.5) is inherently poorly convergent. Because $V_{bx}(r_{bx})$ and $\phi_a(\mathbf{r}_{bx})$ have the same argument, there is no natural cutoff of the r_x integration. To overcome this problem Huby and Mines[6] and Vincent[7] introduce a convergence factor, so that the matrix element in (2.5) is interpreted as

$$\langle\Psi_{xA}^c|\rho_b(\mathbf{r}_x)\Phi_A\rangle\equiv\lim_{\alpha\to0^+}\langle\Psi_{xA}^c|\rho_b^{(\alpha)}\Phi_A\rangle\ , \tag{2.22}$$

$$\rho_b^{(\alpha)}=e^{-\alpha r_x}\rho_b(\mathbf{r}_x)\ . \tag{2.23}$$

For physically possible transitions (i.e., on the energy shell), the r_x dependence of the integrand in the left-hand side (LHS) of (2.22) oscillates with zero mean at large r_x, and therefore the RHS of (2.22) has a well defined limit. A minimum rate of oscillation of the matrix element is determined by the binding energy of ϕ_a. A complete definition of $(\Delta\sigma/\Delta\mathbf{p})_{\mathrm{post}}$ should use the RHS of (2.22) for the matrix element in (2.5). Further discussion of the basis for this convergence procedure appears in Sec. III.

Rather than immediately set $\alpha\to0^+$ in (2.5), we can carry $\alpha\neq0$ through the manipulations that lead to (2.21). With $\alpha\neq0$ the various integrations are finite, and we obtain a reliable, α-dependent breakup fusion cross section

$$^{(\alpha)}\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\mathrm{post}}^{\mathrm{BF}}\equiv\frac{-(2\pi)^4}{\pi v_a}\langle\rho_b^{(\alpha)}|G_x^\dagger W_xG_x|\rho_b^{(\alpha)}\rangle\ . \tag{2.24}$$

We can now inquire whether the $\alpha\to0^+$ limit of the mathematically reliable expression (2.24) equals the BF part of the physical sum (2.5), in which the $\alpha\to0^+$ limit is performed before summation.

Interchange of the order of limits is acceptable if the $\alpha\to 0^+$ limit is uniform with respect to the variables in the subsequent limiting operations. Only two operations are questionable. One is the $\sum_c$ summation over breakup channels of the $x+A$ compound system, which implies integration over the energy sharing and angle dependence of the breakup into two or more particles in various partitions. Although the rate of oscillation of the breakup matrix element at large r_x is affected by these channel variables, the asymptotic wave number of the oscillation is seen to be bounded away from zero by a minimum value, independent of exit channel. Therefore the $\alpha\to 0^+$ limit is uniform with respect to the channel variables in the $\sum_c$ sum.

The other questionable operation is the application of the Green's function G_x to the source function $\rho_b^{(\alpha)}$ in (2.24),

$$\int G_x(\mathbf{r}_x,\mathbf{r}'_x)\rho_b^{(\alpha)}(\mathbf{r}'_x)d^3r'_x , \tag{2.25}$$

which defines an α-dependent function of $\mathbf{r}_x$. This operation would seem to be in doubt in the limit $\alpha\to 0^+$, because in this limit the longest range part of the source function is asymptotically proportional to a plane wave

$$\rho_b(\mathbf{r}'_x)\sim \exp i\mathbf{r}'_x\cdot(\mathbf{k}_a-\mathbf{k}_b) ;$$

here $\mathbf{k}_a$ and $\mathbf{k}_b$ are the asymptotic momenta of the distorted waves. However, the $\alpha\to 0^+$ behavior of (2.25) is controlled by the factor W_x in (2.24), which establishes a finite range for $\mathbf{r}_x$ in (2.25); this tells us that at asymptotically large values of $\mathbf{r}'_x$ the $\mathbf{r}'_x$ dependence of G_x is governed by a definite wave number k_x, determined by $E_x=\hbar^2k_x^2/2m_x$. From the on-energy-shell energetics of breakup we then have $k_x\neq|\mathbf{k}_a-\mathbf{k}_b|$, and therefore the $\alpha\to 0^+$ limit of (2.25) is uniform with respect to $\mathbf{r}_x$. The $\alpha\to 0^+$ limit of (2.24) can therefore be taken inside the scalar product.

We conclude that the mathematical definition of (2.21) as the $\alpha\to 0^+$ limit of (2.24) is sound. The finite imaginary part in (2.11) is correctly separated from the divergent real part.

III. SINGULARITIES OF THE POST-FORM MATRIX ELEMENT

We already noted that the post-form inclusive breakup calculation involves marginal convergence of the individual post-form breakup matrix element

$$T_{\text{post}}=\langle\chi_b^{(-)}\Psi_{xA}^c\,|\,V_{bx}\,|\,\chi_a^{(+)}\phi_a\Phi_A\rangle . \tag{3.1}$$

To exhibit the disconnected part of (3.1) it is sufficient to consider the simple matrix element for elastic breakup

$$T_{\text{post}}^{\text{EB}}=\langle\chi_b^{(-)}\chi_x^{(-)}\,|\,V_{bx}\,|\,\chi_a^{(+)}\phi_a\rangle , \tag{3.2}$$

in which the projection

$$\langle\Psi_{xA}^{c_0}\,|\,\Phi_A\rangle\equiv\langle\chi_x^{(-)}| \tag{3.3}$$

has defined an elastic distorted wave for fragment x, governed by U_x through the equation

$$(K_x+U_x^\dagger)\chi_x^{(-)}=E_x\chi_x^{(-)} . \tag{3.4}$$

In (3.3) the superscript c_0 labels the elastic channel that consists asymptotically of incoming waves plus Φ_A times a plane wave in $\mathbf{r}_x$.

In a configuration space analysis, (3.2) has singularities if for certain combinations of momenta the contributions from asymptotic regions of space are unbounded. The disconnected singularity occurs because $V_{bx}(r_{bx})$ and $\phi_a(\mathbf{r}_{bx})$ have the same argument, therefore there is no natural cutoff of any other, independent variable that enters the integration. This effect is dominated by the longest range part of the integrand, the plane wave part, which we calculate in $\mathbf{r}_{bx},\mathbf{r}_x$ variables by the procedure

$$\int\int d^3r_{bx}d^3r_x e^{-i\mathbf{k}_b\cdot\mathbf{r}_b}e^{-i\mathbf{k}_x\cdot\mathbf{r}_x}V_{bx}\phi_a e^{i\mathbf{k}_a\cdot\mathbf{r}_a}=(2\pi)^3\delta(\mathbf{k}_a-\mathbf{k}_b-\mathbf{k}_x)\int d^3r_{bx}V_{bx}\phi_a\exp i\mathbf{r}_{bx}\cdot\left[\frac{m_b}{m_a}\mathbf{k}_a-\mathbf{k}_b\right] , \tag{3.5}$$

$$\equiv(2\pi)^3\delta(\mathbf{k}_a-\mathbf{k}_b-\mathbf{k}_x)\widetilde{V_{bx}\phi_a} . \tag{3.6}$$

The δ function divergence is suppressed on the energy shell, when a cross section is computed, because momentum conservation from the δ function is not compatible with energy conservation for the breakup products b,x of projectile a. But even on the energy shell, the scattered wave terms in the distorted waves in (3.2) cause additional effects, such as the familiar long range oscillations of the integrand, and these must be dealt with by some limiting procedure.

A more complete treatment of the post-form singularities is obtained by starting with the prior-form matrix element, which has no disconnected parts, and transforming it to post form. The prior-form counterpart of the matrix element in (2.5) is

$$T_{\text{prior}}=\langle\chi_b^{(-)}\Psi_{xA}^c\,|\,U_b+V_{xA}-U_a\,|\,\chi_a^{(+)}\phi_a\Phi_A\rangle . \tag{3.7}$$

Using the Schrödinger equation for the entrance channel, (3.7) becomes

$$T_{\text{prior}}=\langle\chi_b^{(-)}\Psi_{xA}^c\,|\,H_A+V_{xA}+K_x+K_b+V_{bx}+U_b-E_\alpha\,|\,\chi_a^{(+)}\phi_a\Phi_A\rangle , \tag{3.8}$$

where E_α is now the total energy in the entrance channel. The total energy in the exit channel is E_β. On the energy shell $E_\alpha=E_\beta=E$, as in the previous section; however, in the present off-energy-shell discussion we allow $E_\beta\neq E_\alpha$. To recover T_{post} the operators in (3.8) are applied to the

left, taking due note that kinetic energy operators are not Hermitian with respect to continuum wave functions. We obtain

$$T_{\text{prior}}=\langle\chi_b^{(-)}\Psi_{xA}^c \mid V_{bx}+E_\beta-E_\alpha-\overleftrightarrow{K}\mid\chi_a^{(+)}\phi_a\Phi_A\rangle\ , \tag{3.9}$$

$$=T_{\text{post}}+\langle\chi_b^{(-)}\Psi_{xA}^c \mid E_\beta-E_\alpha-\overleftrightarrow{K}\mid\chi_a^{(+)}\phi_a\Phi_A\rangle\ , \tag{3.10}$$

with

$$\langle\psi_1\mid\overleftrightarrow{K}\mid\psi_2\rangle\equiv\langle K\psi_1\mid\psi_2\rangle-\langle\psi_1\mid K\psi_2\rangle\ . \tag{3.11}$$

Again, it is of interest to examine elastic breakup, for which (3.10) becomes

$$T_{\text{prior}}^{\text{EB}}=T_{\text{post}}^{\text{EB}}+\langle\chi_b^{(-)}\chi_x^{(-)}\mid E_\beta-E_\alpha-\overleftrightarrow{K}\mid\chi_a^{(+)}\phi_a\rangle\ . \tag{3.12}$$

Both terms on the RHS of (3.12) lead to oscillatory integrals and off-energy-shell singularities, hence they must be discussed by some kind of limiting procedure.

Since the integrand of T_{prior} is localized within a finite volume of configuration space, one obvious way to control the singularities on the RHS of (3.12) is to restrict all the integrals to a large but finite region of space. By use of Green's theorem the kinetic energy correction term is transformed to a surface integral over the boundary of the integration region. We get

$$T_{\text{prior}}^{\text{EB}}=T_{\text{post}}^{\text{EB}}+\langle\chi_b^{(-)}\chi_x^{(-)}\mid E_\beta-E_\alpha\mid\chi_a^{(+)}\phi_a\rangle$$
$$-\text{surface integral}\ , \tag{3.13}$$

in which it is understood that the remaining volume integrals are limited to the specified bounded region. In the surface integral asymptotic forms of the scattering wave functions

$$\chi_a^{(+)}\to e^{i\mathbf{k}_a\cdot\mathbf{r}_a}+f_a e^{ik_a r_a}/r_a\ , \tag{3.14a}$$

$$\chi_b^{(-)}\to e^{i\mathbf{k}_b\cdot\mathbf{r}_b}+f_b e^{-ik_b r_b}/r_b\ , \tag{3.14b}$$

$$\chi_x^{(-)}\to e^{i\mathbf{k}_x\cdot\mathbf{r}_x}+f_x e^{-ik_x r_x}/r_x\ , \tag{3.14c}$$

can be used. The substitutions (3.14) are inserted simultaneously for all the wave functions in the surface integral, since the factor ϕ_a in (3.12) compels $\mathbf{r}_b$ and $\mathbf{r}_x$ to be simultaneously large.

The plane wave and scattered wave terms of (3.14) make very different contributions to the surface integral. The leading contribution at large radii comes from the product of three plane waves. Again using Green's theorem, this part of the surface integral is transformed back to the plane wave volume integral

$$\langle e^{i\mathbf{k}_b\cdot\mathbf{r}_b}e^{i\mathbf{k}_x\cdot\mathbf{r}_x}\mid\overleftrightarrow{K}\mid e^{i\mathbf{k}_a\cdot\mathbf{r}_a}\phi_a\rangle\ , \tag{3.15}$$

which becomes

$$(E_\beta-E_\alpha)\langle e^{i\mathbf{k}_b\cdot\mathbf{r}_b}e^{i\mathbf{k}_x\cdot\mathbf{r}_x}\mid e^{i\mathbf{k}_a\cdot\mathbf{r}_a}\phi_a\rangle$$
$$+(2\pi)^3\delta(\mathbf{k}_a-\mathbf{k}_b-\mathbf{k}_x)\widetilde{V_{bx}\phi_a}\ , \tag{3.16}$$

using the differential equations for the bra and ket in (3.15), followed by (3.6).

The next most important parts of the surface integral have products of two plane wave terms from (3.14) and one scattered wave. It is seen by calculation that these contributions are finite, and they usually are oscillatory functions of the radius R of the bounding surface, with zero mean. Finiteness is obtained because the scattered wave term decreases as R^{-1}, and because the associated angle integral over a product of two plane waves is proportional to a spherical Bessel function, which contributes another R^{-1} factor. Oscillations occur because, except at unphysical combinations of momenta, energetics constrains the relevant wave numbers not to add to zero. All remaining parts of the surface integral vanish in the limit $R\to\infty$.

The parts of (3.12) that survive at large R are

$$T_{\text{post}}^{\text{EB}}=T_{\text{prior}}^{\text{EB}}+(2\pi)^3\delta(\mathbf{k}_a-\mathbf{k}_b-\mathbf{k}_x)\widetilde{V_{bx}\phi_a}+(E_\alpha-E_\beta)(\langle\chi_b^{(-)}\chi_x^{(-)}\mid\chi_a^{(+)}\phi_a\rangle-\langle e^{i\mathbf{k}_b\cdot\mathbf{r}_b}e^{i\mathbf{k}_x\cdot\mathbf{r}_x}\mid e^{i\mathbf{k}_a\cdot\mathbf{r}_a}\phi_a\rangle)$$
$$+\text{finite surface terms}\ . \tag{3.17}$$

Equation (3.17) displays the structure of the poorly convergent parts of the second term of (3.12), which originate from the non-Hermiticity of the kinetic energy operator in the prior-post transformation. Since $T_{\text{prior}}^{\text{EB}}$ is well behaved, the divergences simply carry through into $T_{\text{post}}^{\text{EB}}$, in the form of the oscillations and off-energy-shell singularities we have previously discussed. Of course the amplitudes have other off-energy-shell singularities, like the well-known "stripping pole," which occur in both T_{post} and T_{prior}. In any case if $E_\beta=E_\alpha$ the divergent terms in (3.17) disappear and the remaining finite surface terms are reliably oscillatory. These oscillatory contributions vanish if the limit $R\to\infty$ is performed using a convergence factor that averages their momenta.[6,7] This well-known averaging is equivalent to the use of wave packets for the incident and observed particles. Let us use a subscript C to indicate use of a convergence factor to suppress oscillations in the calculation of T_{post}. Then the on-energy-shell transformation obtained from (3.17) is

$$T_{\text{post},C}^{\text{EB}}=T_{\text{prior}}^{\text{EB}}\ , \tag{3.18}$$

in agreement with usual practice.

Our specialization to elastic breakup does not limit the analysis of the post-prior transformation. From the elastic analysis we recognize that the second term of (3.10) always vanishes on the energy shell, using momentum averaging. This transformation gives the standard result

$$\langle \chi_b^{(-)}\Psi_{xA}^c \mid V_{bx} \mid \chi_a^{(+)}\phi_a\Phi_A\rangle_C$$
$$= \langle \chi_b^{(-)}\Psi_{xA}^c \mid V_{xA}+U_b-U_a \mid \chi_a^{(+)}\phi_a\Phi_A\rangle \ . \quad (3.19)$$

The transformation also confirms the relation between the particular residual interactions that are appropriate in the post and prior DWBA matrix elements. Off the energy shell the transformation for nonelastic amplitudes is a little different from (3.17), because Ψ_{xA}^c now does not have a plane wave ground state projection. This suppresses the δ function in (3.17).

Although quantum mechanical manipulations with T_{post} may seem to threaten uncontrolled contributions from the above off-energy-shell divergences, the convergence analysis in Sec. II shows that this problem does not affect the sum rule extracted in this paper.

IV. PRIOR-FORM SUM RULE

The general prior-form DWBA inclusive cross section derived from the Hamiltonian (2.1) is

$$\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\text{prior}} = \frac{(2\pi)^4}{v_a}\sum_c \mid \langle \chi_b^{(-)}\Psi_{xA}^c \mid V_\alpha \mid \chi_a^{(+)}\phi_a\Phi_A\rangle \mid^2 \times\delta(E-E_b-E^c) \ , \quad (4.1)$$

with the residual interaction

$$V_\alpha = V_{xA}+U_b-U_a \ . \quad (4.2)$$

The inclusive sum on c begins with the usual steps:

$$\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\text{prior}} = \frac{-(2\pi)^4}{\pi v_a}\,\text{Im}\sum_c \langle \chi_a^{(+)}\phi_a\Phi_A \mid V_\alpha^\dagger \mid \chi_b^{(-)}\Psi_{xA}^c\rangle (E^+-E_b-E^c)^{-1}\langle \Psi_{xA}^c\chi_b^{(-)} \mid V_\alpha \mid \chi_a^{(+)}\phi_a\Phi_A\rangle \ , \quad (4.3)$$

$$= \frac{-(2\pi)^4}{\pi v_a}\,\text{Im}\langle \chi_a^{(+)}\phi_a\Phi_A \mid V_\alpha^\dagger \mid \chi_b^{(-)}\rangle (E^+-E_b-H_{xA})^{-1}(\chi_b^{(-)} \mid V_\alpha \mid \chi_a^{(+)}\phi_a\Phi_A\rangle \ . \quad (4.4)$$

We rewrite (4.4) as

$$\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\text{prior}} = \frac{-(2\pi)^4}{\pi v_a}\,\text{Im}\langle \chi_a^{(+)}\phi_a \mid B \mid \chi_a^{(+)}\phi_a\rangle \ , \quad (4.5)$$

where

$$B \equiv (\Phi_A \mid V_\alpha^\dagger \mid \chi_b^{(-)})(E^+-E_b-H_{xA})^{-1}(\chi_b^{(-)} \mid V_\alpha \mid \Phi_A) \quad (4.6)$$

is introduced for later discussion. In the above equations we note particularly that $V_\alpha^\dagger \neq V_\alpha$, because the optical potentials in V_α are not Hermitian. Furthermore, V_{xA} in V_α depends on the internal coordinates of the target nucleus, and this prevents an immediate optical reduction of the Green's function in (4.4), as in Sec. II.

At a corresponding stage of analysis Udagawa and Tamura[10] argue that V_α is primarily responsible for projectile breakup, that target nucleus excitation tends to occur as a "second step." This physical model suggests an approximation in which V_α is replaced by its ground state expectation

$$\bar{V}_\alpha = \bar{V}_{xA}+U_b-U_a \equiv (\Phi_A \mid V_{xA} \mid \Phi_A)+U_b-U_a \ . \quad (4.7)$$

Insertion of $\bar{V}_\alpha$ for V_α in (4.4) allows Φ_A to pass through the matrix elements to the Green's function, so that the optical reduction of (2.10) can be applied. The resulting approximate prior-form sum rule becomes

$$\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\text{prior}}^{\text{LUT}} = \frac{-(2\pi)^4}{\pi v_a}\,\text{Im}\langle \rho_a \mid G_x \mid \rho_a\rangle \ , \quad (4.8)$$

with a source function

$$\rho_a = (\chi_b^{(-)} \mid \bar{V}_{xA}+U_b-U_a \mid \chi_a^{(+)}\phi_a\rangle \ . \quad (4.9)$$

This approximate expression is identically the starting point of the LUT paper.

In our analysis the effect of V_{xA} in V_α is calculated exactly by rearranging (4.4) before carrying out the ground state expectation. Using (2.4) we rewrite V_α as

$$V_\alpha = (H_{xA}+E_b-E^+)+(U_b-U_a-H_A-K_x-E_b+E^+) \ , \quad (4.10)$$

so that the first term of (4.10), which incorporates V_{xA}, commutes with $\chi_b^{(-)}$ and combines with the Green's function. The second term of (4.10) commutes with Φ_A. The ground state expectation in (4.6) then yields

$$B = (U_b^\dagger-U_a^\dagger-K_x+E_x) \mid \chi_b^{(-)})(E_x^+-K_x-U_x)^{-1}(\chi_b^{(-)} \mid (U_b-U_a-K_x+E_x)$$
$$+ \mid \chi_b^{(-)})(K_x-\bar{V}_{xA}-E_x)(\chi_b^{(-)} \mid + (U_a^\dagger-U_b^\dagger) \mid \chi_b^{(-)})(\chi_b^{(-)} \mid + \mid \chi_b^{(-)})(\chi_b^{(-)} \mid (U_a-U_b) \ , \quad (4.11)$$

where again $E_x = E-E_b-E_A$. As before, the formal optical potential U_x in (4.11) is generated by the optical reduction. Equation (4.11) is converted to a more familiar form by another step of rearrangement, like (4.10), to eliminate (E_x-K_x) from the numerator of the first term. We get

$$B = (U_b^\dagger-U_a^\dagger+U_x) \mid \chi_b^{(-)})(E_x^+-K_x-U_x)^{-1}(\chi_b^{(-)} \mid (U_b-U_a+U_x)+ \mid \chi_b^{(-)})(U_x-\bar{V}_{xA})(\chi_b^{(-)} \mid \ . \quad (4.12)$$

It is convenient to express (4.12) as

$$B=(U_b+U_x-U_a)^\dagger\,|\,\chi_b^{(-)}\rangle(E_x^+-K_x-U_x)^{-1}\langle\chi_b^{(-)}\,|\,(U_b+U_a-U_x)$$
$$+2iW_x\,|\,\chi_b^{(-)}\rangle(E_x^+-K_x-U_x)^{-1}\langle\chi_b^{(-)}\,|\,(U_b+U_x-U_a)+\,|\,\chi_b^{(-)}\rangle(U_x-\overline{V}_{xA})\langle\chi_b^{(-)}\,|\ . \tag{4.13}$$

Substitution in (4.5) gives the exact prior-form breakup sum rule,

$$\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\rm prior}=\frac{-(2\pi)^4}{\pi v_a}(\mathrm{Im}\langle\rho_a'\,|\,G_x\,|\,\rho_a'\rangle$$
$$+2\,\mathrm{Re}\langle n\,|\,W_xG_x\,|\,\rho_a'\rangle$$
$$+\langle n\,|\,W_x\,|\,n\rangle)\ , \tag{4.14}$$

with

$$|\,n(\mathbf{r}_x)\rangle\equiv(\chi_b^{(-)}\,|\,\chi_a^{(+)}\phi_a\rangle\ , \tag{4.15}$$

$$|\,\rho_a'(\mathbf{r}_x)\rangle\equiv(\chi_b^{(-)}\,|\,U_b+U_x-U_a\,|\,\chi_a^{(+)}\phi_a\rangle\ . \tag{4.16}$$

To get the third term of (4.14) we note that $\overline{V}_{xA}$ is real, so $\mathrm{Im}(U_x-\overline{V}_{xA})=W_x$.

Although the first term of (4.14) resembles the LUT sum rule, the remaining terms are new. But even the first term differs from LUT, since ρ_a' contains U_x where in their ρ_A LUT have $\overline{V}_{xA}$.

Further understanding of (4.14) is obtained by the successive application of two identities. In the first step transformation (2.18) is applied to the first term of (4.14), to extract from it the inclusive elastic breakup (EB) cross section. The remaining inclusive BF cross section then is

$$\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\rm prior}^{\rm BF}=\frac{-(2\pi)^4}{\pi v_a}(\langle\rho_a'\,|\,G_x^\dagger W_xG_x\,|\,\rho_a'\rangle$$
$$+2\,\mathrm{Re}\langle n\,|\,W_xG_x\,|\,\rho_a'\rangle$$
$$+\langle n\,|\,W_x\,|\,n\rangle)\ . \tag{4.17}$$

The relation between (4.17) and the post-form BF cross section of (2.21) is obtained with a second identity,

$$G_x\,|\,\rho_a'\rangle=G_x\,|\,\rho_b\rangle-\,|\,n\rangle\ , \tag{4.18}$$

related to one used by LUT. Insertion of (4.18) in (4.17) immediately yields (2.21). Thus *the post and prior expressions for inclusive breakup are equivalent,* both at the level of the individual many-body matrix elements of (2.5) and (4.1), and at the level of the formal sum rules (2.21) and (4.17).

We examine the identities used above: Transformation (2.18) decomposes the prior-form inclusive cross section into two parts

$$\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\rm prior}=\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\rm prior}^{\rm EB}+\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\rm prior}^{\rm BF}\ , \tag{4.19}$$

of which the BF part is given already as (4.17). The EB part of (4.14) is obtained as[3]

$$\left[\frac{\Delta\sigma}{\Delta\mathbf{p}}\right]_{\rm prior}^{\rm EB}=\frac{-(2\pi)^4}{\pi v_a}\langle\rho_a'\,|\,(1+G_x^\dagger U_x^\dagger)\,\mathrm{Im}G_0(1+U_xG_x)\,|\,\rho_a'\rangle$$
$$=\frac{(2\pi)^4}{v_a}\sum_{\mathbf{k}_x}|\,\langle\chi_b^{(-)}\chi_x^{(-)}(\mathbf{k}_x)\,|\,U_x+U_b-U_a\,|\,\chi_a^{(+)}\phi_a\rangle\,|^2\delta\left[E_x-\frac{k_x^2}{2m_x}\right]\ . \tag{4.20}$$

It is important to recognize (4.20) as the correct, physical elastic-breakup cross section. At a more basic level, the many-body DWBA matrix element for elastic breakup is known to be

$$\langle\chi_b^{(-)}\Psi_{xA}^{c_0}\,|\,V_{xA}+U_b-U_a\,|\,\chi_a^{(+)}\phi_a\Phi_A\rangle\ , \tag{4.21}$$

where c_0 labels the elastic channel, as in (3.3). We simplify (4.21) by the familiar exact relation

$$\langle\Psi_{xA}^{c_0}\,|\,V_{xA}\,|\,\Phi_A\rangle=\langle\chi_x^{(-)}\,|\,U_x\ , \tag{4.22}$$

and obtain (4.20). Use of $\overline{V}_{xA}$ in (4.20) instead of U_x would produce an incorrect expression for the EB cross section.

Identity (4.18) formally resembles a post-prior transformation given by LUT. We present an independent derivation, both because of physical differences in the interactions used, and to examine a possible role of surface terms. Thus, cross multiplication and subtraction of the differential equations

$$(K+V_{bx}+U_a-E)\chi_a^{(+)}\phi_a=0\ , \tag{4.23a}$$

$$(K+U_b+U_x-E)G_x(\mathbf{r}_x,\mathbf{r}_x')\chi_b^{(+)}(-\mathbf{k}_b,\mathbf{r}_b)$$
$$=-\delta(\mathbf{r}_x-\mathbf{r}_x')\chi_b^{(+)}(-\mathbf{k}_b,\mathbf{r}_b)\ , \tag{4.23b}$$

gives

$$G_x(\chi_b^{(-)} \mid V_{bx}+U_a-U_b-U_x-\vec{\vec{K}} \mid \chi_a^{(+)}\phi_a\rangle = \mid n\rangle \ , \tag{4.24}$$

where we recall

$$\chi_b^{(-)*}(\mathbf{k}_b,\mathbf{r}_b)=\chi_b^{(+)}(-\mathbf{k}_b,\mathbf{r}_b) \ .$$

Equation (4.24) is equivalent to (4.18), with an extra non-Hermiticity correction $\vec{\vec{K}}$. It was already seen in Sec. III that the $\vec{\vec{K}}$ term can be converted into a surface integral, whose value depends on details of the continuum wave functions. One of the wave functions in (4.24) is $G_x(\mathbf{r}_x,\mathbf{r}'_x)$, an asymptotically outgoing function of $\mathbf{r}'_x$ if $\mathbf{r}_x$ is finite, with no asymptotic plane wave part. With one wave function of that structure, the surface integral must be finite and oscillatory. It vanishes if any slight averaging of momenta is performed, as by one of the familiar convergence procedures. We conclude that (4.18) is justified. It is also clear in (4.24) that the prior interaction $U_b+U_x-U_a$ in ρ'_a transforms into the post interaction V_{bx} in ρ_b.

V. CONCLUSIONS

The most immediate conclusion is that the post and prior inclusive breakup theories are equivalent, both at the level of the initial summed cross section expressions and at the level of closed-form ground state expectations. Despite a variety of interesting complications in the post-form analysis, the key to equivalence is the correction of several approximations in the basic prior-form sum rule derived by Udagawa and Tamura[10] and used by LUT. These corrections should cause improvements in the comparisons with experimental data discussed by those authors.

As is usual with sum rules, the simplicity of the closed-form expressions, especially in post form, is a little deceiving. In the present case we must keep in mind that U_x in our derivation is a complicated formal operator, as in (2.16), both nonlocal and a fluctuating function of energy. In practical applications the energy average[2] of U_x is replaced by the empirical optical potential for particle x. Too much should not be expected. Also it should not be forgotten that the sum rule derivations ignore excitations of the target nucleus by the spectator particle b, though of course the effect of virtual excitations of the target on the motion of b is included in U_b.

Several basic lessons about reaction theory emerge. One concerns the two-step physical model emphasized by LUT, who picture breakup and target nucleus excitation as disjoint events. We see that this heuristic model does correspond to the mathematical formulation of the post-form calculation, but not of the prior-form calculation; in the latter case *excitation during breakup* substantially alters the sum rule. Thus, our *qualitative description of the reaction depends on the formalism used,* an interesting reversal of usual expectations.

The importance of excitation during breakup should be no surprise. In the prior interaction V_{xA} is symmetrical in its relation to the nuclei a,A. If it disrupts the structure of one of them, we must suppose it simultaneously has an equally strong effect on the other.

We are also reminded that a reduced theory of a many-body system, in terms of the coordinates of only a few "active" particles, requires the careful introduction of effective interactions. In the present case the many-body operator

$$V_\alpha = V_{xA}+U_b-U_a$$

is not reduced to the coordinates of particles b and x merely by calculating its expectation with respect to Φ_A. We must instead ask *how* V_α enters in the original many-body theory. The analysis generates several reduced terms; these contain the effective operator U_x. Another example of this concern appears in the prior-form matrix element for elastic breakup, discussed in connection with (4.17)—(4.19). Here again the many-body matrix element (4.18) is not reduced merely by replacing V_{xA} by the folded interaction $\overline{V}_{xA}$. Again in this simple case the correct reduced replacement for V_{xA} is the optical potential U_x.

ACKNOWLEDGMENTS

We are grateful to Professor T. Tamura and Professor T. Udagawa for prepublication copies of the LUT paper and for further discussions and correspondence. We thank Professor T. Terasawa for valuable comments about the problems of surface integrals and about convergence. N.A. is grateful to the Japan Society for the Promotion of Science for a fellowship that led to the collaboration with M.I. Nuclear theory research at the University of Pittsburgh was supported by the National Science Foundation.

APPENDIX: ON THE LUT CRITIQUE OF THE POST-FORM SUM RULE

The methods of the present paper are easily applied to examine the Appendix in which LUT criticize our previous papers: For example, from our analysis in Sec. III we see that several surface integrals neglected by LUT indeed make vanishing contributions. However, we find a serious flaw in the discussion following their Eq. (A10). Introducing the standard DWBA replacement

$$\Psi^{(+)} \rightarrow \chi_a^{(+)}\phi_a\Phi_A \ ,$$

Eq. (A10) takes the more explicit form

$$[\rho_b(\mathbf{r}_x)+\alpha(H_{xA}-E_x)n(\mathbf{r}_x)] \mid \Phi_A\rangle \ . \tag{A10'}$$

Equation (A10′) is understood to contain a convergence factor for the coordinate r_x, discussed in Sec. II B of the present paper, as is required in the post-form matrix element. Omitting unnecessary coefficients, substitution of (A10′) in (A5) of LUT gives

$$\begin{aligned}&\mathrm{Im}\langle\rho_b \mid (E_x^+-H_x)^{-1} \mid \rho_b\rangle \\ &\quad -\mathrm{Im}(\langle\alpha n \mid \rho_b\rangle+\langle\rho_b \mid \alpha n\rangle \\ &\qquad +\langle\alpha n\Phi_A \mid H_{xA}-E_x \mid \alpha n\Phi_A\rangle) \ .\end{aligned} \tag{A6'}$$

The derivation of (A6′) uses the algebraic reduction of a product of many-body operators, as in the steps from (4.6) to (4.11) of the present paper. The quantity in parentheses would diverge if the convergence factor in (A10′) were omitted; on the other hand this quantity is real, hence its imaginary part vanishes for all values of the convergence parameter. Thus (A6′) reduces to the previous post-form sum rule of (A6) of LUT, in agreement with our papers. There is no α-dependent ambiguity. Equation (A11) of LUT and the discussion based on it are wrong.

[1]X.-H. Li, T. Udagawa, and T. Tamura, Phys. Rev. C **30**, 1895 (1984).
[2]N. Austern and C. M. Vincent, Phys. Rev. C **23**, 1847 (1981).
[3]A. Kasano and M. Ichimura, Phys. Lett. **115B**, 81 (1982).
[4]For a review of this work see G. Baur, F. Rösel, D. Trautmann, and R. Shyam, Phys. Rep. **111**, 333 (1984).
[5]C. M. Vincent and H. T. Fortune, Phys. Rev. C **8**, 1084 (1973).
[6]R. Huby and J. R. Mines, Rev. Mod. Phys. **37**, 406 (1965).
[7]C. M. Vincent, Phys. Rev. **175**, 1309 (1968).
[8]H. Feshbach, Ann. Phys. (N.Y.) **5**, 357 (1958); **19**, 287 (1962).
[9]K. Masutani and K. Yazaki, Phys. Lett. **104B**, 1 (1981).
[10]T. Udagawa and T. Tamura, Phys. Rev. C **24**, 1348 (1981).